Chapter 3.7: Implicit Derivation

Explicit and Implicit Functions

Explicit function

$$y = \boxed{\text{stuff with x}}$$

It is clear how to compute y from x.

Implicit function

$$\boxed{\text{y is mixed with x} = \text{other mix}}$$

For a given x, y has to satisfy something but not always clear what that is.

Example: Which of the following are implicit and explicit functions?

$$y = x^2 + \sin\left(\tan\left(x^2\right)\right)$$

$$x^2 + y = e^x + 17$$

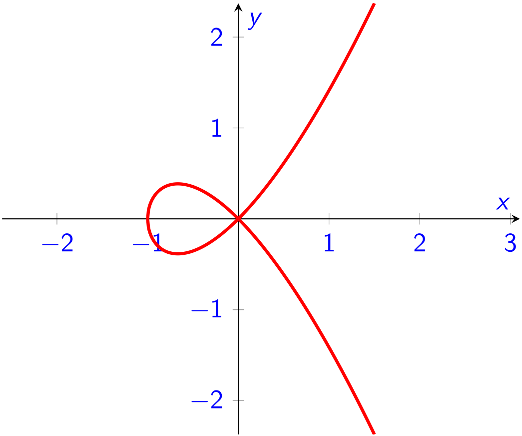
$$\sin\left(x + e^y\right) = x^2 - \cos(y) - y$$

$$y = 5t^2 + \cos(t) - e^t$$

$$x^2 + y^2 = 25$$

$$s = 5y^2 - \cos y$$

Not All Curves are Functions of x



$y^2 = x^2(x+1)$

Goal: Be able to find tangents to these curves.

Derivative of Equal Functions

Observation:

If $f(x) = g(x)$, then $f'(x) = g'(x)$.

Example: Let $f(x) = \sin(2x)$ and $g(x) = 2\sin(x)\cos(x)$. Verify that $f'(x) = g'(x)$.

$f'(x) = \cos(2x) \cdot 2 = 2\cos(2x)$

Recall:

$\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$

$g'(x) =$
$2\cos(x)\cos(x) - 2\sin(x)\sin(x) =$
$2\cos(2x)$

Notice that indeed $f'(x) = g'(x)$.

Main idea: Take the derivative of both side according to x. Think of y as a function of x and apply the chain rule whenever appropriate.

Example:

$$x^2 + y^2 = 25$$
$$x^2 + (y(x))^2 = 25$$

Take derivative of both sides of the equation according to x.

$$x^2 + y^2 = 25$$
$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[25\right]$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Implicit Differentiation

1. Start with relationship between x and y and take the derivative of both sides with respect to x.
2. Apply the chain rule carefully for y. Any derivative of y becomes $\frac{dy}{dx}$.
3. After taking the derivatives, rearrange carefully to solve for $\frac{dy}{dx}$.

Evaluation $\frac{dy}{dx}$ at (a, b):

$$\left.\frac{dy}{dx}\right|_{(a,b)}$$

Example: Find $\left.\frac{dy}{dx}\right|_{(1,1)}$ given that

$$x^2 + xy - y^2 = 1$$

$$\frac{d}{dx}\left[x^2 + xy - y^2\right] = \frac{d}{dx}\left[1\right]$$

$$2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$2x + y + \frac{dy}{dx}[x - 2y] = 0$$

$$\frac{dy}{dx} = \frac{2x + y}{2y - x}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2 \cdot 1 + 1}{2 \cdot 1 - 1} = 3$$

Tangent and Normal Lines

Example: Find tangent line to implicitly defined curve at $(2, 4)$:

$$x^3 + y^3 = 9xy$$

$$\frac{d}{dx}\left[x^3 + y^3\right] = \frac{d}{dx}\left[9xy\right]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$

The slope is $\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{3 \cdot 4 - 2^2}{4^2 - 3 \cdot 2} = 0.8$

$y = 0.8x + b$. Hence $4 = 0.8 \cdot 2 + b$ and $b = 2.4$, Tangent is $y = 0.8x + 2.4$.

A *normal* line to a curve at a point is a line passing through the point perpendicular to the tangent line.

Example: Find normal line to implicitly defined curve $x^3 + y^3 = 9xy$ at $(2, 4)$.

$y = \frac{-1}{0.8}x + b$ and $\frac{1}{0.8} = \frac{10}{8} = \frac{5}{4}$

So $y = -\frac{5}{4}x + b$

Plug in $(2, 4)$ and we get

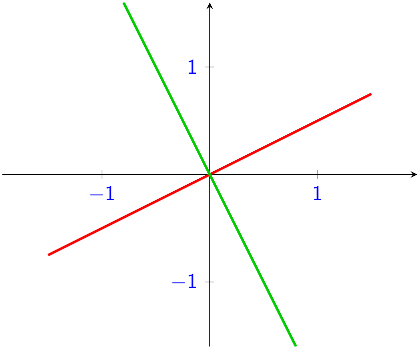
$$4 = -\frac{5}{4} \cdot 2 + b$$

$$b = 4 + \frac{5}{2} = \frac{13}{2}$$

Solution is $y = -\frac{5}{4} + \frac{13}{2}$

Normal Line Intermezzo

Find normal line $y = kx$ to $y = ax$ at $(0,0)$. Example has $a = 0.5$



Make triangle with points $(0,0),(1,0),(1,a)$. Then take perpendicular triangle. The triangle has points $(0,0),(-a,0),(-a,1)$. The point $(-a,1)$ is on $y = kx$, so solve $1 = -ak$ and get $k = \frac{-1}{a}$.

What is the slope of the normal line to line $y = ax + b$ if $b \neq 0$? also $k = \frac{-1}{a}$.

Second Derivatives

Example: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $3x^3 = 6 + 2y^2$.

First we compute $y' = \frac{dy}{dx}$.

$$\frac{dy}{dx}\left[3x^3\right] = \frac{dy}{dx}\left[6 + 2y^2\right]$$
$$9x^2 = 0 + 4y \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{9x^2}{4y} = y' \text{ for } y \neq 0$$

Now the second derivative

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left[\frac{9x^2}{4y}\right] = \frac{18x4y - 9x^2 \cdot 4\frac{dx}{dy}}{(4y)^2} = \frac{18x}{4y} - \frac{36x^2}{(4y)^2} \cdot \frac{dx}{dy}$$
$$= \frac{9x}{2y} - \frac{36x^2}{(4y)^2} \cdot \frac{9x^2}{4y} = \frac{9x}{2y} - \frac{324x^4}{(4y)^3} \text{ for } y \neq 0$$

Chapter 3.7 Recap

1. Take the derivative of both sides with respect to x.
2. Apply the chain rule carefully for y. Any derivative of y becomes $\frac{dy}{dx}$.
3. After taking the derivatives, solve for $\frac{dy}{dx}$.

A normal line is perpendicular to the tangent line.